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Aspects of Solutions of Massive Spin-3/2 Equation in Schwarzschild Space-Time

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Abstract The general massive spin-(3/2) (Rarita–Schwinger) field equation in Schwarzschild geometry, previously separated by variable separation, is further studied. The orthogonality of the solutions of the angular equations is exploited. The study of the radial equations, that are proposed in the most detailed form, is reduced to the study of four coupled differential equations. The equations are discussed and integrated near the Schwarzschild radius and for zero and large values of the radial coordinate. A covariant product of states is considered that is induced by a conserved current. It is shown the existence of states that are bound in the scalar product without implying the existence of a discrete energy spectrum.

Keywords Schwarzschild geometry · Massive spin 3/2 equations · Solution · Product of states

1 Introduction

Recently it has been shown that the general field equations of arbitrary spin can be separated by an elementary variable separation method in the Schwarzschild space-time [9]. The result has been obtained by an induction based on the explicit study for spin 3/2 field equation. The case is of physical interest because, in a general curved space-time, the general spin 3/2 field equation can be interpreted in term of Rarita–Schwinger field equation of spin 3/2 (see, e.g. [4]; for a formulation including torsion see, e.g. [10]).

Due to the interest of the argument, it is the object of the present paper to further develop the study of the separated spin 3/2 field equations in the Schwarzschild space-time. Previous

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results are obtained again with improvements. The explicit form of the solutions of the angular equation are determined and their orthogonality relations given. For what concerns the separated radial equations they are formulated in the completely detailed form. There results the study of the solution of a system of 10 coupled differential equations. The equations are reduced, by substitution, to the study of 4 coupled radial differential equations. These equations are integrated near the origin of the radial coordinate, near the Schwarzshild radius and at infinity. The solutions are put in a form where some global aspects are evident. By using a conserved current, a covariant product of states is formulated in a standard way. The positivity of the scalar product is not ensured a priori. By using the properties of the solutions of the radial equations previously obtained, it is proved the existence of states that are bound in the product. The result does not imply the existence of discrete value of the energy.

2 Separation of the Equation

The general spin-3/2 massive field equation relative to the spinors $\phi_{ABC} = \phi_{(ABC)}$, $\theta_{ABX'} = \theta_{BAX'}$ can be written in a general curved space-time as [4, 5]:

$$\nabla^{D}_{X'}\phi_{DA_1A_2} + i\mu_\star\theta_{A_1A_2X'} = 0, \tag{1a}$$

$$\nabla^{X'}_{(A}\theta_{A_1A_2)X'} - i\mu_\star\phi_{AA_1A_2} = 0, \tag{1b}$$

$\mu_\star\sqrt{2} = m_0$ the mass of the field particles. The equation has been separated [9] in the Schwarzschild space-time of line element

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin\theta^2 d\varphi^2) \tag{2}$$

by means of the Newman–Penrose formalism [6] and by choosing as null tetrad frame the one employed in Chandrasekhar's book, [2]. The separation can be obtained by the variables separation, by setting

$$\begin{aligned}
&\phi_{ABC} = \phi_h(r)S_h(\theta)F(t,\varphi), \quad F = e^{im\varphi+ikt}, \quad h = 0, 1, 2, 3,\\
&\theta_{000'} = F(t,\varphi)S_1(\theta)\theta_{00'}(r), \qquad \theta_{001'} = -F(t,\varphi)S_0(\theta)\theta_{01'}(r),\\
&\theta_{010'} = F(t,\varphi)S_2(\theta)\theta_{10'}(r), \qquad \theta_{101'} = -F(t,\varphi)S_1(\theta)\theta_{11'}(r),\\
&\theta_{110'} = F(t,\varphi)S_3(\theta)\theta_{20'}(r), \qquad \theta_{111'} = -F(t,\varphi)S_2(\theta)\theta_{21'}(r)
\end{aligned} \tag{3}$$

where it has been set $h = A + B + C = 0, 1, 2, 3$. One can also assume $m = 0, \pm 1, \pm 2, \ldots$.

The separated angular equations can be reduced to the eigenvalue equations

$$\begin{aligned}
L^{+}_{-1/2}L^{-}_{3/2}S_0 &= \lambda_1\lambda_4 S_0,\\
L^{+}_{1/2}L^{-}_{1/2}S_1 &= \lambda_2\lambda_5 S_1,\\
L^{-}_{1/2}L^{+}_{1/2}S_2 &= \lambda_2\lambda_5 S_2,\\
L^{-}_{-1/2}L^{+}_{3/2}S_3 &= \lambda_3\lambda_6 S_3,
\end{aligned} \tag{4}$$

$L_d^{\pm} = \partial_\theta \mp m \csc\theta + d \cot\theta$. The separation constants λ_i, $i = 1, 2, \ldots, 6$, are subjected to the consistency condition $\lambda_1\lambda_4 = \lambda_3\lambda_6 = \lambda_2\lambda_5 + 1 \equiv -\lambda^2$ [8, 9]. If one looks for the solutions S_i having essentially a polynomial form, one finds that the possible values of $\lambda^2 > 0$ are

$$\lambda^2 = l(l+1) - \frac{3}{4}, \quad l = |m|+1, |m|+2, \ldots. \tag{5}$$

(The expression of the constant of λ^2 was erroneously reported in [8, 9].) In case $m \geq 0$ one obtains the solutions $S_{0lm}(\theta) = (1-\xi)^{\frac{m}{2}+\frac{3}{4}}(1+\xi)^{\frac{m}{2}-\frac{3}{4}} P_{l-m}^{(m-\frac{3}{2}, m+\frac{3}{2})}(\xi)$, $S_{1lm}(\theta) = (1-\xi)^{\frac{m}{2}-\frac{1}{4}}(1+\xi)^{\frac{m}{2}+\frac{1}{4}} P_{l-m}^{(m-\frac{1}{2}, m+\frac{1}{2})}(\xi)$, where $\xi = \cos\theta$ and $P_n^{(\alpha,\beta)}$ are the Jacobi Polynomials [1]. (If $m < 0$ the differential equations for S_0, S_1 are the same of those with $m > 0$ if one replaces m by $|m|$ and ξ by $-\xi$.) Moreover $S_{2lm} \equiv S_{1l-m}$, $S_{3lm} \equiv S_{0l-m}$. The angular functions $S^{(h)}(\theta, \varphi) = S_{hlm}(\theta)e^{im\varphi}$, $h = 0, 1, 2, 3$, can then be assumed to be already ortho-normalized

$$\int d\Omega\, S_{lm}^{(h)}(\theta,\varphi)(S_{l'm'}^{(h)}(\theta,\varphi))^\star = \delta_{mm'}\delta_{ll'}, \quad h = 0, 1, 2, 3. \tag{6}$$

By taking into account the symmetrization in (1b), the 10 separated radial equations result to have the detailed form ($\phi' = d\phi/dr$)

$$\left(\frac{ikr}{r-2M} + \frac{3}{r}\right)\phi_1 + \phi_1' - i\mu_\star\theta_{00'} = \frac{\lambda_1}{r\sqrt{2}}\phi_0, \tag{7}$$

$$\left(\frac{ikr}{r-2M} + \frac{2}{r}\right)\phi_2 + \phi_2' - i\mu_\star\theta_{10'} = \frac{\lambda_2}{r\sqrt{2}}\phi_1, \tag{8}$$

$$\left(\frac{ikr}{r-2M} + \frac{1}{r}\right)\phi_3 + \phi_3' - i\mu_\star\theta_{20'} = \frac{\lambda_3}{r\sqrt{2}}\phi_2, \tag{9}$$

$$\left(\frac{ik}{2} - \frac{M+r}{2r^2}\right)\phi_0 + \frac{2M-r}{2r}\phi_0' - i\mu_\star\theta_{01'} = \frac{\lambda_4}{r\sqrt{2}}\phi_1, \tag{10}$$

$$\left(\frac{ik}{2} + \frac{3M-2r}{2r^2}\right)\phi_1 + \frac{2M-r}{2r}\phi_1' - i\mu_\star\theta_{11'} = \frac{\lambda_5}{r\sqrt{2}}\phi_2, \tag{11}$$

$$\left(\frac{ik}{2} + \frac{7M-3r}{2r^2}\right)\phi_2 + \frac{2M-r}{2r}\phi_2' - i\mu_\star\theta_{21'} = \frac{\lambda_6}{r\sqrt{2}}\phi_3, \tag{12}$$

$$\left(\frac{ikr}{r-2M} + \frac{1}{r}\right)\theta_{01'} + \theta_{01'}' - i\mu_\star\phi_0 = -\frac{\lambda_4}{r\sqrt{2}}\theta_{00'}, \tag{13}$$

$$\frac{2ikr}{r-2M}\theta_{11'} + 2\theta_{11'}' + \left(\frac{ik}{2} - \frac{5M-r}{2r^2}\right)\theta_{00'} + \frac{2M-r}{2r}\theta_{00'}' - 3i\mu_\star\phi_1$$
$$= -\frac{1}{r\sqrt{2}}(2\lambda_5\theta_{10'} + \lambda_1\theta_{01'}), \tag{14}$$

$$\left(\frac{ikr}{r-2M} - \frac{1}{r}\right)\theta_{21'} + \theta_{21'}' + \left(ik - \frac{M}{r^2}\right)\theta_{10'} + \frac{2M-r}{r}\theta_{10'}' - 3i\mu_\star\phi_2$$
$$= -\frac{1}{r\sqrt{2}}(\lambda_6\theta_{20'} + 2\lambda_2\theta_{11'}), \tag{15}$$

$$\left(\frac{ik}{2}+\frac{3M-r}{2r^2}\right)\theta_{20'}+\frac{2M-r}{2r}\theta'_{20'}-i\mu_\star\phi_3=-\frac{\lambda_3}{r\sqrt{2}}\theta_{21'}. \tag{16}$$

The study of the solutions of the radial equations can be reduced to the study of four coupled differential equation obtained by inserting the expressions of the θ spinor of (7–12) into (13–16). One obtains:

$$\frac{3\sqrt{2}\lambda_4}{r(r-2M)}\phi_1=\phi''_0+\frac{2r+M}{r(r-2M)}\phi'_0+\phi_0\left[\frac{k^2r^2+3ikM}{(r-2M)^2}-\frac{M+\lambda^2r+m_0^2r^3}{r^2(r-2M)}\right], \tag{17}$$

$$\begin{aligned}\frac{\sqrt{2}\lambda_5\phi_2}{r(r-2M)}-\frac{\lambda_1\phi_0}{\sqrt{2}r^2}&\\ =\phi''_1+\frac{2r-M}{r(r-2M)}\phi'_1+\phi_1\Bigg[&\frac{k^2r^2}{(r-2M)^2}+\frac{11M-r10/3}{r^2(r-2M)}\\ &+\frac{ikM}{(r-2M)^2}-\frac{\lambda^2+2/3+m_0^2r^2}{r(r-2M)}\Bigg],\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\sqrt{2}\lambda_6\phi_3}{r(r-2M)}-\frac{\sqrt{2}\lambda_2\phi_1}{r^2}&\\ =\phi''_2+\frac{2r-3M}{r(r-2M)}\phi'_2+\phi_2\Bigg[&\frac{k^2r^2}{(r-2M)^2}+\frac{11M-r10/3}{r^2(r-2M)}\\ &-\frac{ikM}{(r-2M)^2}-\frac{\lambda^2+2/3+m_0^2r^2}{r(r-2M)}\Bigg],\end{aligned} \tag{19}$$

$$-\frac{3\lambda_3\phi_2}{\sqrt{2}r^2}=\phi''_3+\frac{2r-5M}{r(r-2M)}\phi'_3+\phi_3\left[\frac{k^2r^2-3ikM}{(r-2M)^2}-\frac{M+\lambda^2r+m_0^2r^3}{r^2(r-2M)}\right]. \tag{20}$$

These equations are difficult to be solved exactly. Some properties of the solutions are discussed in the following sections.

3 Local and Global Aspects of the Solutions of the Radial Equations

Indications about the solutions of the radial equations can be obtained by formally setting $\lambda^2=0$. On account of the consistency conditions of the angular separation constants, one can set $\lambda_1=\lambda_3=\lambda_4=\lambda_6=0$, $\lambda_2\lambda_5=-1$ in the radial equations. Then a possible solution of the system (17–20) is given by $\phi_1=\phi_2=0$ while ϕ_0, ϕ_3 satisfy decoupled Fuchs type equations whose behavior in $r=0$, $r=2M$, $r=\infty$ can be easily determined.

It is a fact that for $\lambda\neq0$ there are solutions of the system (17–20) that behave in the critical points similarly to the case $\lambda^2=0$, as it suggested by the following remarks. If one considers the limit $r\to\infty$ of the right hand sides of (17–20), one can check that they all assume the same asymptotical differential expression. Moreover, by equating to zero the right hand sides of (17–20), one finds that all the associated indicial equations admit the root $-2ikM$. This support the representation of the solutions of (17), (20) in the factorized form

$$\phi_j=r^{\alpha_j}(r-2M)^{-2ikM}f_j,\quad j=0,1,2,3. \tag{21}$$

For $r \to \infty$ the expressions (21) are solutions of the system (17–20) provided

$$
\begin{aligned}
&f_j \stackrel{r\to\infty}{\longrightarrow} B_j r^{\alpha'_j} \exp[\pm i r\sqrt{k^2 - m_0^2}],\\
&\alpha'_j = -(\alpha_j - 2ikM + 1) \pm i\frac{M(2k^2 - m_0^2)}{\sqrt{k^2 - m_0^2}},\\
&3\sqrt{2}\lambda_4 B_1 = -B_0[\alpha'_0(\alpha'_0 - 1) + 2\alpha'_0(\alpha_0 - 2ikM)],\\
&2\lambda_5 B_2 - \lambda_1 B_0 = \sqrt{2}B_1[\alpha'_1(\alpha'_1 - 1) + 2\alpha'_1(\alpha_1 - 2ikM)],\\
&\sqrt{2}\lambda_6 B_3 - \sqrt{2}\lambda_2 B_1 = B_2[\alpha'_2(\alpha'_2 - 1) + 2\alpha'_2(\alpha_2 - 2ikM)],\\
&3\lambda_3 B_2 = -\sqrt{2}B_3[\alpha'_3(\alpha'_3 - 1) + 2\alpha'_3(\alpha_3 - 2ikM)].
\end{aligned}
\tag{22}
$$

In this context it is understood that the constant B_i, λ_i obey to the seven bounds given by the above equations for the B_i's and the mentioned consistency conditions for the λ_i's. Possible values of the α_k's are given below.

For $r \to 2M$ the expressions (21) are solutions of the system (17–20) provided $\phi_j = A_j(r - 2M)^{-2ikM}$, $A_j = \text{const}$, $j = 0, 1, 2, 3$, with

$$
\begin{aligned}
&3\sqrt{2}\lambda_4 A_1 = -A_0(2M)^{\alpha_0-\alpha_1-1}[M + 2M\lambda^2 + 8M^3 m_0^2],\\
&\sqrt{2}\lambda_5 A_2 = A_1(2M)^{\alpha_2-\alpha_1}[3/2 - \lambda^2 - 4M^2 m_0^2],\\
&\sqrt{2}\lambda_6 A_3 = A_2(2M)^{\alpha_2-\alpha_3}[3/2 - \lambda^2 - 4M^2 m_0^2].
\end{aligned}
\tag{23}
$$

For $r \to 0$ the expressions (21) are possible asymptotic solutions provided

$$
\begin{aligned}
&\alpha_0 = \alpha_1 = -\alpha_3 = 1, \quad \alpha_2 = 0,\\
&f_0 = C_0 + D_0 r, \qquad f_1 = C_1, \qquad f_2 = C_2, \qquad f_3 = C_3 r + D_3
\end{aligned}
\tag{24}
$$

with the constants C_i, D_i satisfying the constraints

$$
\begin{aligned}
&3\sqrt{2}\lambda_4 C_1 + \lambda^2 C_0 - 3MD_0 = 0,\\
&\sqrt{2}\lambda_5 C_2 + \sqrt{2}\lambda_1 MC_0 - 10MC_1 = 0,\\
&\sqrt{2}\lambda_6 D_3 - 11MC_2 = 0,\\
&3\sqrt{2}\lambda_3 MC_2 + \lambda^2 D_3 + MC_3 = 0.
\end{aligned}
\tag{25}
$$

One can note that for $k^2 < m_0^2$ there are solutions that decay exponentially for large r. This suggests to explore whether in this case there are bound solutions under some to be defined product of states. The argument is of interest also in view of a quantization of the scheme. A possible approach to this problem is given by the following considerations.

4 Product of States

A standard way to define a covariant product of solutions of (1) is that of considering the current

$$
J^{AX'}(\psi, \psi') = i[\phi^A_{BC}\bar{\chi}^{X'BC} + \theta_{BC}{}^{X'}\bar{\xi}^{ABC} - \text{c.c.}]
\tag{26}
$$

with $\psi = (\phi, \theta)$ solution of (1) and $\psi' = (\xi, \chi)$ solution of the system complex conjugate of the system (1). The current is conserved, $\nabla_{AX'} J^{AX'} = 0$, provided the spinors satisfy the field equations [4]. Accordingly the expression

$$(\psi, \psi') = \int |g|^{\frac{1}{2}} \sigma^{\alpha}_{AA'} J^{AA'}(\psi, \psi') n_\alpha d\Sigma \tag{27}$$

$$\equiv \frac{1}{\sqrt{2}} \int d\Omega \int_0^\infty dr r^2 \left(\sqrt{\frac{r}{r-2M}} J^{00} + \sqrt{\frac{r-2M}{r}} J^{11} \right) \tag{28}$$

defines a properly covariant scalar product (e.g., [3]). The Cauchy surface Σ has been chosen to be $t = t_0$ and the unit vector orthogonal to Σ has been chosen to be $n^\alpha \equiv (\sqrt{\frac{r}{r-2M}}, 0, 0, 0)$. In the tetrad adopted here to separate (1), (namely the one of [2]), one has $\sigma^t_{AA'} = \frac{1}{\sqrt{2}} \operatorname{diag}\{\frac{r}{r-2M}; \frac{1}{2}\}$. By further choosing

$$\bar{\chi}^{X'AB}_{k'l'm'} \equiv \theta^{ABX'}_{-k'l'-m'}, \qquad \bar{\xi}^{ABC}_{k'l'm'} \equiv \phi^{ABC}_{-k'l'-m'} \tag{29}$$

(note that $\bar{\phi}_{-k}(\bar{\lambda}_i) = \phi_k(\lambda_i), \bar{\theta}_{-k}(\bar{\lambda}_i) = -\theta_k(\lambda_i)$) the angular part of the integral (28) can be carried out by using the positions (4) and equation (6), so that one is left with

$$(\psi, \psi) \simeq \int_0^\infty dr r^2 \Bigg[\sqrt{\frac{r}{r-2M}} \Im(\theta_{21}\bar{\phi}_1 - 2\theta_{11}\bar{\phi}_2 + \theta_{01}\bar{\phi}_3) - \frac{1}{2}\sqrt{\frac{r-2M}{r}} \Im(\theta_{20}\bar{\phi}_0 - 2\theta_{10}\bar{\phi}_1 + \theta_{00}\bar{\phi}_2) \Bigg] \tag{30}$$

where it is understood that all spinors have the same labels k, l. If now one considers the behaviors (21), (23), (24) of ϕ_h, and the corresponding behaviors of $\theta_{hX'}$ as obtained from (7–12), one can check that the integral (30) converges both in $r = 0$ and $r = 2M$. Therefore, from (22), for $k^2 < m_0^2$, the integral (30) exists finite.

5 Remarks

In the previous Sections the behavior of the radial solutions of the spin 3/2 field equation in Schwarzschild space-time has been determined near the critical points $r = 0$, $r = 2M$, $r = \infty$. The result enables to show, for $k^2 < m_0^2$, the existence of bound states in the scalar product induced by the conserved current. The fact holds also for spin 0 and spin 1 in the Schwarzschild metric as it can be proved by applying the previous considerations to the results of Zecca [7, 11]. This is a peculiar character because in the conventional quantum mechanics such states would correspond to a discrete spectrum as, e.g., the energy spectrum of the hydrogen atom. Instead in the present scheme there is no trace of discrete k^2 values.

It must be noted that positivity of the scalar product is not automatically verified here. A further study of the problem does not seem easy. On the one hand an exact solution of (17–20) is difficult to be obtained. On the other hand the $\theta_{hX'}$ appearing in the integral (30) are not simply proportional, as in the Robertson–Walker space-time case [8], to the ϕ_i's, but have the more complicated expression in term of the ϕ's given by (7–12).

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